COLOR-SATURATION AND ITS QUANTITATIVE RELATIONS.

By A. Kirschmann.

Without considering their dependence on time and space, our sensations of sight are variable in three directions:

I. In the strength of the sensation as light, that is, in brightness or *light-intensity*.

II. In the quality of sensation, that is, in its color or *color-tone*.

III. In the strength in which the special color-tone is perceived independently of its light-intensity, that is, in its *color-saturation*.

The first two of these variables are commonly understood and evident. The third is not generally recognized, and the necessity of its introduction is disputed. It requires, therefore, a further explanation.

If all qualities of light were coördinate, there would be no necessity for introducing a third variable, but since one of the qualities, that of the uncolored (so-called white) light, plays an exceptional rôle, we are not able to represent all possible light-sensations by a two-dimensional system or by an expression with two variables. All those qualities commonly known as colors or color-tones form a closed manifoldness, in which there is a gradual transition from any particular color to any other. All qualities in this manifoldness are coördinate, and every one can be regarded as a transition between its neighbors ; *e. g.*, the different tones of green form the transition between yellow and blue, as blue is the transition between green and violet, and so with all colors. The transitions are integral parts of the whole manifoldness, and together constitute it.

All these qualities are variable in intensity. If we represent the manifoldness of color-tones by a straight or curved line of any form and length, we must represent the system of all possible degrees of intensity of these various light qualities by a plane or curved surface. This state of affairs is not changed even if we regard uncolored light as a coördinate quality and give it and its degrees of intensity a place with the others on that surface. But it will be easily seen that

thereby the transitions between uncolored light and the various colors are not included; for between uncolored light and every color-tone there are transitional qualities, which are neither colorless nor one of the color-tones of the above mentioned manifoldness. In order to give all transitions of this kind a place on the two-dimensional representation, we should have to interpose uncolored light between every two neighboring color-tones, which would contradict the conditions upon which this surface was constructed.

Hence, if we are to represent by a space construction all possible color-sensations, including these transitions between colorless light and the various colors, we must make use of the third dimension, or, in other words, we must introduce the third variable. This third variable of light-sensation, according to which there is an infinite or at least a great number of transitions between any color-tone and colorless light, constitutes what we call, after Wundt, "the saturation of light sensation." Three-dimensional constructions which take into account this third variable are the *Color-pyramid*, first commended by Lambert;[1] the *Double-pyramid* and the *Color-sphere*, both devised by Runge;[2] and the Color-cone or Double-cone described by Wundt.[3]

In passing we may be allowed to interpolate some remarks on the physical theory of colors and a suggestion concerning a correction of the color-cone.

1. It is said that the color which forms the transition between red and violet, purple, is not a simple color, but a mixture of the two ends of the spectrum. On the contrary we hold that purple is not only as a sensation just as simple as any other color-tone, but that it even has the same right as other colors to be called a constituent part of the white light. We are guided in these assumptions by the following reasons:

In a ray of white light all wave-lengths are in the same space; every one coöperates at every point of the ray with every other, especially with those which are, as regards wave-length and vibration-number, its nearest neighbors. In a linear spectrum the wave-lengths of the extremes are deprived of their right to act together. The ends of the spectrum therefore stand under changed conditions. It is true this will have no influence if the color is really a function of the

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[1] Beschreibung einer mit Calan'schen Wachse gemalten Farbenpyramide, Berlin, 1872.

[2] Die Farbenkugel oder Construction des Verhältnisses aller Mischungen der Farben zu einander, Hamburg, 1810.

[3] Wundt, Vorl. üb. Menschen- und Thierseele, p. 114. Phys. Psychologie, 4th ed., I Band, p. 503.

wave-length, which nearly everybody to-day takes for granted, but which nevertheless is wrong, or at least uncertain. It is claimed that light of one certain wave-length causes the impression of red, another wave-length that of green, etc.; but we have that only from hypothesis, for *nobody has ever seen light of one wave-length.*

Helmholtz, to whom we are so greatly indebted for his works on optics, has pointed out that the purity of a spectrum stands in reciprocal relations to the width of the slit of the spectroscope, but he did not draw the proper inferences from this proposition, for he does not object to neglecting the width of the slit as something extremely small. It is true the width of the slit of a spectroscope or the distance between the lines of a grating can be reduced to a small fraction of a millimeter, and it is then indeed very small compared with the objects surrounding us; but it represents still a space of considerable extension compared with wave-lengths and vibration-numbers. Suppose we have obtained a spectrum of one meter in length by means of a slit one hundredth of a millimeter wide. Then we shall have at any point of the spectrum not one wave-length only, but a superposition of many slit-images, each corresponding to a certain wave-length. It is easily seen that the difference of the wave-lengths or vibration-numbers, which coöperate at one point of the spectrum, depends on the width of the slit. In the case supposed above the vibration-number increases in the distance of a meter, that is, from the red to the violet end (if we accept 412,000,000,000,000 as the vibration-number of the least, and 790,000,000,000,000 as that of the most refrangible ray which is visible) by 378,000,000,000,000 vibrations. If the change of vibration-number were equally distributed over the whole spectrum, the change in a part of it corresponding to the width of the slit would be $\frac{1}{100000}$ of the total change, *i. e.*, 3,780,000,000 vibrations. That same number will indicate the difference of the rays which coöperate at one point of this comparatively very pure spectrum; for on every point of it there will not be one wave-length, but many, the smallest and the largest of which differ by not less than 3,780,000,000 vibrations in a second. Since the increase of vibration-numbers is not equally distributed over the whole spectrum, the number which expresses the degree of superposition of rays will be different in the different regions of the spectrum, and it is very likely that the variation of the superposition, which does not necessarily follow the same constant as the variation of wave-lengths, has something to do with the periodicity in the manifoldness of color-sensations.

Since we have no right to neglect magnitudes such as the above mentioned, we cannot longer claim that the color-quality is a function of the wave-length. It may just as well be—and the probability for this supposition is even greater—that the color-quality is a function of the *superposition of wave-lengths*, so that to every qualitative difference in spectral colors corresponds a difference in the mode of the superposition. A further consideration shows that this superposition does not take place at the ends of the spectrum to the same degree as in the other spectral regions. At any other place in the spectrum a wave-length coöperates with its neighbors on both sides. At the extremes the rays are deprived of this possibility, the coöperation here being confined to one side only. These changed conditions at the ends of the spectrum are of course not a quality of the rays of light concerned, but a peculiarity of that special method used to separate the components of white light. If we place the red end of the spectrum and the violet end, which has about double the number of vibrations, under the same conditions as other colors, *i. e.*, if we put them together in such a way that they are neighbors and partly overlap each other, we shall see the color-tone *purple*, which is missing in the spectrum. These conditions are met in Newton's Rings, in some cases of anomalous dispersion and in the following very simple experiment. If we invert the ordinary spectroscopic arrangement, where the source of light is a bright line on dark ground and view a narrow black surface (*e. g.*, a strip of black velvet) on a bright background through a prism, we shall see a kind of inverted spectrum with red-purple in the middle. We must agree that the existence of this color does not prove anything more than that the mixture of the ends of the spectrum gives purple, but the spectrum just referred to has another remarkable property. If we regard the extremes of it, we find at the one end yellow and at the other end blue, or since the term blue is rather flexible, we may say a somewhat greenish blue. But the green proper is entirely missing in this spectrum. We think the theory according to which the color is a direct function of the wave-length cannot give a satisfactory account of this phenomenon, which can easily be explained by the above stated theory of superposition. In the experiment mentioned it is another part of the spectrum which is deprived of the possibility of the proper superposition of wave-lengths, viz., the green, which therefore must disappear, as the purple does in the ordinary spectrum. It is for the very same reason that the series of interference-colors as they are seen in the polarization-microscope does not begin, as it should according to the

theory, with the green, but with the yellow of first order.

I may mention in this connection that the inverted spectrum referred to above can be projected on a screen by means of a lantern just as well as the ordinary spectrum. In order to do this, it is only necessary to replace the usual slit by a glass plate, of which a little square or oblong part is made opaque by covering it with black paint or paper. The colors obtained on the screen by this method are, provided that the "negative slit" has the proper extension and is correctly focused, just as brilliant as those of a positive spectrum derived from the same source of light; and the objection that they were not pure enough to admit any conclusions about the cause of the absence of the green, can be proved unfounded by the following simple experiment, which I do not find reported in the optical literature known to me.

When light, reflected from a very thin sheet of mica, is examined with the spectroscope, there appear in the spectrum a number of rather sharply defined black stripes, which are caused by the interference of two components of the light; the one of these components is reflected from the front surface, the other from the back surface of the mica. The way-difference implies for all wave-lengths, for which the distance of the two reflecting surfaces is not a multiple of $\frac{\lambda}{2}$, a phase-difference also and thus gives rise to interference. The number of the stripes depends of course on the thickness of the mica. If the sheet is thick, then the distance between the two surfaces is, for many of the wave-lengths, a multiple of $\frac{\lambda}{4}$, and the spectrum will show a greater number of black bands.[1] If the sheet is very thin, it satisfies this condition for a few wave-lengths only, and the number of interference-bands will be limited. It is quite possible to split the mica to such a thickness that the spectrum shows only two or three bands. I even succeeded in obtaining films which caused only one interference-band; in this case the film appears colored for the naked eye. (The apparent color is of course complementary to that one which is extinguished in the spectrum.) These interference-bands can be nicely projected on the

[1]Lewis Wright describes in his book, "Light, a Course on Experimental Optics," a similar experiment, but the figures by which he illustrates it can not be correct for in a dispersion-spectrum, which his illustration apparently represents, the distance between the bands should increase from the red to the violet end. The distance of these interference bands, which can be obtained so easily and with so simple means, might serve well for determining the wave-lengths.

screen, if arrangements are made, so that the light before passing the slit is reflected from the mica-film. Now, if we use, instead of a slit, the above mentioned arrangement for producing an inverted spectrum, the interference-bands can be seen distinctly. This could not take place if the colors in this case were not comparatively pure.

2. It is a mistake to place at the ends of the axis of the color-sphere or double-cone "white" and "black," for these expressions do not designate correctly the extremes of the achromatic series of light sensations. Black and white are not simple sensations, but presentations of a rather complicated composition. As soon as all knowledge about the surrounding objects, the shape and other properties of the surface concerned is excluded, we are not able to determine whether a surface is black, gray or white,—we see only colorless light of more or less intensity. In my lectures I am accustomed to demonstrate this by a very simple but instructive experiment. Black, white and gray surfaces are shown to an observer, who looks through a tube, which is blackened inside and provided with diaphragms so as to confine the visible field to about a square inch. Under these conditions the observer, who is compelled to judge simply according to his sensation of light, is not able to distinguish black, white and gray. If he is a careful thinker, who endeavors to avoid ambiguous terms where they cannot be defined, he will only see more or less intense colorless light. If he is not so careful about his expressions, or if he has not had an opportunity to examine the bearing of the terms black, white and gray, he will mix these "qualities" up. He then sees a piece of white paper, when badly illuminated, "almost black or very dark gray;" and he proclaims a sheet of black cardboard which receives the full daylight as perfectly "white."

The very same primitive apparatus may be used to demonstrate that "brown" is not a simple color-quality. An observer who was requested to determine the colors of the surfaces which he saw through the tube, and to pay special attention to brown colors, if they occurred, never made a mistake in discerning red, green, yellow, blue, violet, purple and their transitions, but he saw a dark chocolate-brown as "rose or pink," and a very dark coffee-brown as "yellow." He was shown various tones and shades of brown, but looking through the tube he always called them "red," "orange" or "yellow."

According to the foregoing consideration, we should place at the ends of the double-cone not black and white, but the minimum and maximum intensity of the achromatic series

of light sensations, or the threshold and upper limit. We may designate these limits in our further discussions and in our figure as the colorless light sensations of the intensities 0 and ∞.

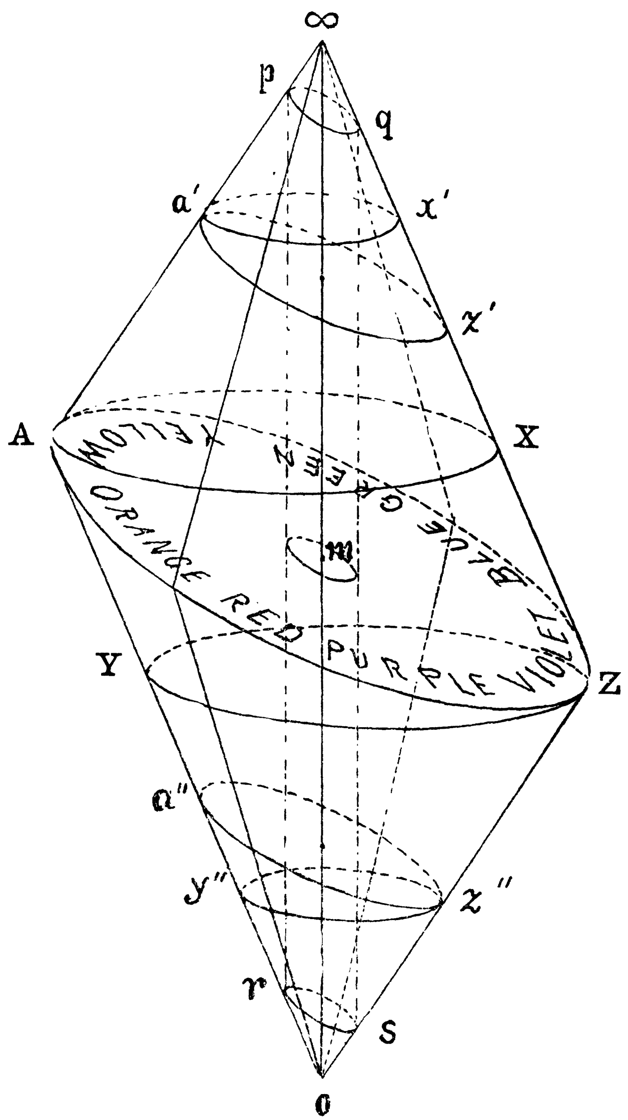

FIG. 1.

In our geometrical representations of the system of possible light and color-sensations, the series of color-tones in their highest saturation is arranged along the equator of the sphere, and in the cone along the circumference of the base. We think it necessary to give up the normal relation of this circle to the axis and to introduce an inclined position of the same as it is represented in Fig. 1. At that point of the circle which is nearest to that end of the axis which represents the maximum intensity of uncolored light sensations, we must place the brightest colors, yellow and yellow-green, and at the other end of the same diameter, *i. e.*, at that point where the circle approaches nearest to that end of the axis, which represents the minimum light-intensity, there will then be the darkest colors, blue and violet.

It is known as a fact that the manifoldness of color-tones decreases with the approach to the extremes of light intensity, and therefore the geometrical representation is not a cylinder, but a cone or a sphere. As long as we do not know the quantitative relations of this decrease, the shape of the surface, spherical, conical or otherwise, and the length of the axis are irrelevant to the theory. The same may be said to a certain degree of the angle of inclination of the color circle to the axis. On a sphere this circle will have such a position with regard to the axis as on a celestial globe the ecliptic has with regard to the equatorial axis.

Since in a sphere the length of the axis is determined by the circumference, while in the system of our light-sensations there is no direct relation between the extension of the chromatic and achromatic series, the cone, which leaves greater play to the latter, must be considered as the more adequate representation.

In the double cone with inclined position of the base are represented the three variables of light-sensations and their relations to one another. From apex to apex and parallel to this direction, we have all possible degrees of light intensity. Around the base of the cone we have all possible color-tones, and from the surface to the axis we find all possible transitions in saturation, from the maximum saturation which the intensity in that case allows, to the saturation zero, *i. e.*, colorless light.

Every plane section through the axis, *e. g.*, orange-∞-blue-0 in Fig. 1, will show a surface which contains all intensities and saturation-degrees of a pair of complementary colors, the axis forming the uncolored neutral line between them.

Every section at right angles to the axis, *e. g.*, $A\ X$ or $Y\ Z$, represents a surface with all the different color-tones and various saturations, but of equal light-intensity. Cylindrical sec-

tions whose axes coincide with that of the cone, represent surfaces of equal saturation, but of different colors and intensities.

That change of color, saturation being constant, is only within certain limits independent of intensity, is in our construction expressed by the fact that the cylindrical surfaces which represent equal saturation become shorter, the more they approach the base of the cone, where their length becomes equal to zero. That is, in other words: If I wish to go from a yellow of highest saturation to a red of equally high saturation, I cannot do this without lowering the intensity. The violet which is of the same saturation as the most saturated yellow, must be of lower intensity, and the violet which is in intensity equal to the most saturated yellow, possesses a lesser degree of saturation, while on the other hand a yellow which corresponds in brightness to the best saturated violet, must necessarily be of much less intensity and saturation than the best saturated yellow. The only pair of complementary colors which have their maximum saturation at equal intensities must be at the ends of that diameter of the base which stands at right angles to the axis; this will be satisfied somewhere near red and blue-green.

The color-cone with inclined base takes into account also the Phenomenon of Purkinje, *i. e.*, the dependence of the color-tone on the light-intensity. For every color-quality we have to assume a double threshold, a threshold of light-intensity, and a threshold of color-intensity, *i. e.*, saturation, and the two must be dependent on each other. A color, in order to be seen in its characteristic quality, must have a certain brightness as light, and the strength of the color as such must have reached a certain degree. If we assume that the saturation-threshold would be the same for all colors and intensities, we may represent this threshold by a cylinder $p\ q\ r\ s$ in our figure, which surrounds concentrically the axis of the double cone at a certain distance. Now it will be easily seen that this cylinder cuts the surface of the upper half of the cone in a circle, which has its highest point, that is, that point which is nearest to the apex, at the yellow or orange. That means, if we increase the light-intensity of the colors, yellow and orange will be the last to lose their special color-quality. At the lower part of the cone, which represents the decrease of intensity, the opposite takes place. Here the blue and violet colors have an advantage, while, in decreasing intensity, red and orange are the first to lose their characteristic color-tone. But the Phenomenon of Purkinje shows still another aspect of the dependence of quality on intensity. All colors, when their intensity is highly increased, show a

tendency to approach in their quality towards orange or yellow, while with considerable decrease of the light-intensity a notable change towards the blue or violet-blue can be observed. This is demonstrated in our color-cone by the circumstance that the sections of equal intensity have in all cases but one, an eccentric position to the axis. If a section at right angles to the axis, representing equal intensities, be made at the middle-point m, it will easily be seen that the centre of gravity of this surface coincides with the point m. If a section be made at any place in the upper part of the cone, the centre of gravity will not lie in the axis, but will be eccentric towards the yellow; and the degree of eccentricity will depend on the distance of the section from the middle point of the system, *i. e.*, on the intensity. Thus, *e. g.*, in the section $a' x'$ of our figure, the whole manifoldness of colors possible at that intensity, is changed in such a way that its greater part is on the side of the red, orange, yellow. The opposite is the case with sections in the lower half of the double cone, *e. g.*, $y'' z''$, where the centre of gravity for every section of equal intensity is eccentric towards the blue or violet, and in consequence of this the manifoldness of colors is moved towards these qualities.

There are still some writers on the subject of color-sensations who seem to hold that saturation, *i. e.*, the third variable of light-sensation, is an illegitimate and unnecessary invention. The adherents of the component-theories, who cannot yet rid themselves of the logical error that simplicity must be not only a useful principle for representation, but also a necessary factor for explanation, show a certain tendency to regard a threefold variability of light sensation as already too complicated. Hering, in order to account for the Phenomenon of Purkinje, introduces the "white-valence" as a property of color. But here, as on many other occasions, this excellent author confuses the psychical qualities of immediately given facts and the products of inferences drawn from other sources. If "white-valence" is a property of color-*sensation*, then we must be able to perceive it as something different from light-intensity, which is not the case. On the other hand, if "white-valence" is a physical property of light, like the energy of waves or the mode of polarization, then it does not stand in any direct relation to color-sensations, and cannot serve as a means of explanation for their relations.

I think those who still hold that it is possible to produce all variations of light-sensations by means of color-tone and light-intensity, can be persuaded of the erroneousness of their assumption by a single and decisive experiment. If we were

able to produce a surface which would have in all its parts the same color-tone and the same light-intensity, and yet show differences in its appearance, there could be no further objection raised to the introduction of the third variable of light-sensation, called saturation (or degree of color or intensity of color-tone). In the remainder of this paper we shall report a method by means of which such surfaces can be obtained.

Every light or color-sensation is simple, although the corresponding physical stimulus may be more or less complex. This is admitted even by adherents of the component theories. Müller, in his recently published article on the psychophysics of visual sensation, distinguishes fundamental and mixed-sensations, but declares both to be simple.[1] The saturation of a color-sensation depends (besides its

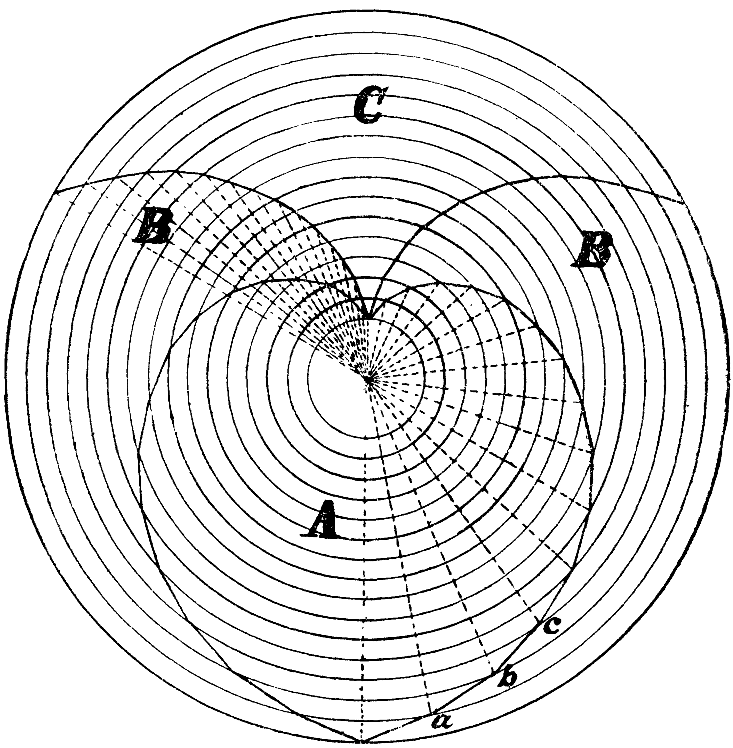

FIG. 2.

[1] *Zeitschrift für Psychologie und Physiologie der Sinnesorgane.* Vol. X, p. 34.

dependence on the constitution and disposition of the sense-organ, and on the co-existing contents of consciousness) on the composition of the physical stimulus. We have seen that "color" is produced by superposition of wave-lengths. The saturation of the color is the greater,the smaller the difference between the superposed wave-lengths is. The simplest way to diminish the saturation of a color is to mix it with colorless light. This can be easily done by means of rotating discs. In order to show on the same surface all transitions from the saturation zero of a color to the highest saturation, which we can obtain in pigment-colors, we have to construct a disc similar to that shown in Fig. 2.

Within a circle of the size of the desired disc, we draw a great number of concentric circles of gradually increasing radius and at equal distance from each other. Then we draw from the middle point and at equal angular distances just as many radii as we have concentric rings. In our figure we have fifteen rings ; therefore we have to divide the angle of 180 degrees into fifteen equal parts, making thus the space between the radii each = 12°. If we then connect the points where the radii meet their corresponding circles (*i. e.*, the points where the first radius and the first circle, the second radius and the second circle intersect) by a curve and draw symmetrically to it on the other side of the diameter the same curve, we shall have the heart-shaped leaf *A* in Fig. 2. The curves so obtained divide the circles in such a way that the part of the circle which lies outside of the leaf is always proportional to the distance of that circle from the innermost one. In other words, the arcs inside of the leaf *A* decrease towards the periphery in an arithmetical progression, while the outside arcs beginning at the innermost circle increase in the same manner.

Now if we cut out of the colored paper a leaf of the form *A* in our figure, paste it on the gray, black or white disc and rotate the so prepared disc with sufficient speed, we shall see the fully saturated color in the centre and colorless light at the circumference, and between centre and circumference all transitions from one extreme to the other. On the other hand, if we cut the leaf out of white, black or gray paper and take a colored disc as the ground, this arrangement, when in rotation, will show the same transitions as above, but in inverted order, that is, the greatest saturation at the circumference, the minimum at the centre.

But with this method we do not arrive at pure saturation-degrees ; there is always complication by the change of intensity. If we use black, we get all shades ; if we apply white, all tints of the color concerned. If we wish to avoid

these complications by the application of gray, there remains the question: which of all possible grays between black and white must we take? In order to get rid of all differences of intensity, we ought to mix the color at every place of the disc with that gray which has exactly the same light intensity as the color itself. In order to accomplish this condition, we have first to ascertain the light-intensity of the color concerned. The photometry of colored light is not an easy matter; but there have been applied several methods which lead to satisfactory results. There is a very ingenious method devised by Prof. Rood.[1] Another way to find out the light intensity of colored surfaces the reader will find in my article on the quantitative relations of light and color-contrast.[2] Every determination of the brightness of a color is of course, on account of the Phenomenon of Purkinje, valid only for a certain constant intensity of the illumination.

Suppose we have by means of our photometrical method ascertained that for a certain intensity of the illumination, which we must keep constant, the color of the leaf *A* as regards its light-intensity is equal to a gray composed of x° white and $360-x^\circ$ black. In order to mix this gray with all the different quantities of the color, we must divide the arcs outside of the leaf *A* (Fig. 2) according to the ratio $\frac{x}{360-x}$ (For our figure we have made the simple assumption that this fraction was $=\frac{1}{2}$.) By this procedure we separate the part *C* of the disc, and this part is to be covered with white, while the remaining part *B* must be covered with black. In our figure we have drawn only fifteen circles. If we use all the circles possible between the edge of the disc and that point from which we wish to start the change of saturation, the curves which limit the leaf and the white portion of the disc will be Archimedic spirals, which are for polar-coördinates what the straight line is for rectangular-coördinates. If we set the so constructed disc in rotation, we shall see the same color-tone and the same intensity on the whole surface, but the saturation decreases from the centre to the circumference in an arithmetical progression. We may briefly state the mathematical deduction of our curve.

The conditions to be satisfied are that the saturation starts at a certain distance *d* from the centre and decreases in such a way that the length of the radius *r* and the arc of the

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[1] Rood, "On a Photometric Method which is Independent of Color." *American Journal of Science*, XLVI, Sept., 1893, p. 173.

[2] Kirschmann, "Ueber die quantitativen Verhältnisse des simultanen Helligkeits- und Farben-Contrastes." *Philos. Studien*, Vol. VI, p. 463 ff.

corresponding angular value of the color-component, ϕ, are inversely proportional; or:

When $r = d$, $\phi = 180°$;
" $r = a + d$, $\phi = 180 - a$;
" $r = na + d$, $\phi = 180 - na$;

from which follows the equation of the curve

$$\phi = 180 - (r - d)\,\frac{a}{a};$$

and if we put $\frac{a}{a} = \mu$,

$$\phi = 180\,(r - d)\,\mu. \quad (1)$$

The value of μ is dependent on the size of the disc. If we wish to have the saturation 0 at the distance R from the centre, we must satisfy the special condition that

$$\phi = 0°, \ r = R.$$

The above stated equation (1) takes, then, the form

$$\mu\,(R - d) = 180,$$

$$\text{or } \mu = \frac{180}{R - d}.$$

If we substitute this value for μ in the equation of the curve, we have

$$\phi = 180 - \frac{r - d}{R - d}\,180,$$

$$\text{or } \phi = 180\,(1 - \tfrac{r-d}{R-d}) \quad (2)$$

We have to determine now the equation for the curve which divides the remainder of the disc into a white and a black part. Suppose the intensity of the color was equal to that of a gray composed of $n°$ white and $m°$ black. The ratio of the white sector to the whole surface left by the color, then, will be $\frac{n}{n+m}$. And since the angular value of the whole uncolored surface must at any distance from the centre be $180°-\phi$, the angular value of the white always will be

$$(180° - \phi)\,\frac{n}{n+m};$$

or if we substitute the above stated value for ϕ,

$$\frac{180\,(r-d)}{R-d} \cdot \frac{n}{n+m}. \quad (3)$$

In order to eliminate possible errors introduced by the spatial arrangement, it will be advisable to carry out each series of experiment with two discs, the one of the above stated arrangement, the other with the saturation increasing

from the centre to the periphery. In this case the equations, corresponding to the above stated (2) and (3), read as follows:

$$\phi = \frac{180\,(r-d)}{R-r},$$

and the angular value of the white sector, the width of which is now decreasing from the centre to the periphery, can be expressed thus:

$$180\,\left(1 - \frac{r-d}{R-d}\right) \cdot \frac{n}{m+n}.$$

In order to make the parts of the discs blend at a smaller speed of the rotation-apparatus, we need only to replace the coefficient 180 in the expressions by an aliquot part of it, *e. g.*, by 60. The construction is then carried out for every third instead of for the whole disc. The discs represented in Figs. 3 and 4 are obtained in this way.

These discs allow the application of the psychophysical methods to the quantitative investigation of color-saturation. We shall, however, confine ourselves in the following to the discussion of the circumstances under which we found the application of the method of average errors practicable and successful.

A disc of either kind described (Fig. 3 or 4) was rotated by an electromotor with a speed of about thirty turns per second. At a certain constant distance in front of the disc on a pedestal was fastened in a horizontal position a graduated circle, which carried a small telescope provided with a spirit-level. The telescope was movable in a horizontal and vertical direction, and its axis, when horizontal and in the position zero, pointed exactly at the centre of the disc. By means of the spirit-level purely azimuthal movements could be secured.

Attached to the telescope was a vernier, which permitted the reading to be made down to minutes. The vernier was a double one, connected at both sides of the telescope, so that every position could be ascertained by two readings. The lenses of the telescope were removed and the field of the instrument was so small that no differences of saturation could be detected within it. To the eye-piece was attached a small screen, which prevented the observer from seeing any other part of the disc except that surface which was shown through the telescope. In Fig. 5, which gives an illustration of the whole apparatus, this screen is not represented.

The method of our experiment was the following. The observer sees through the telescope, without knowing its position, a certain saturation, and after the position is changed, he is asked to find, by moving the telescope, the same saturation again. This can be sought, of course, on the same

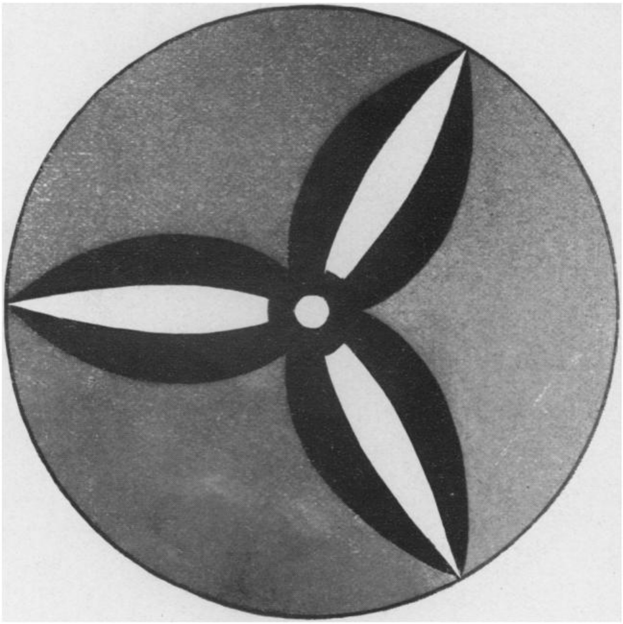

Fig. 3.

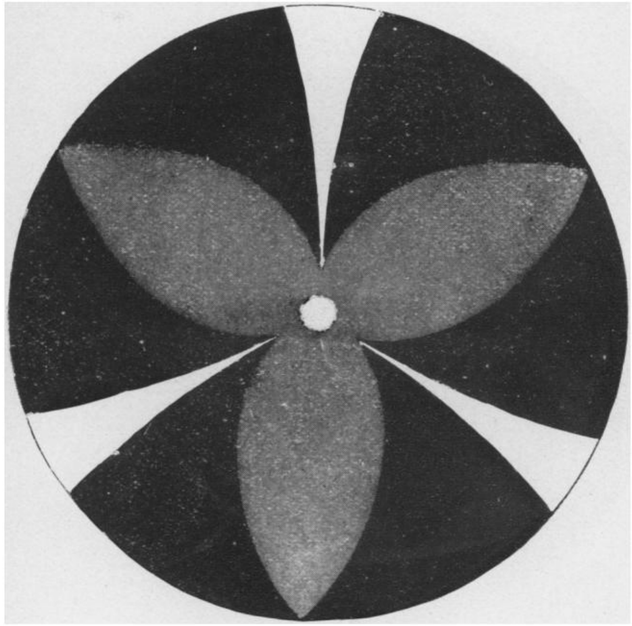

Fig. 4.

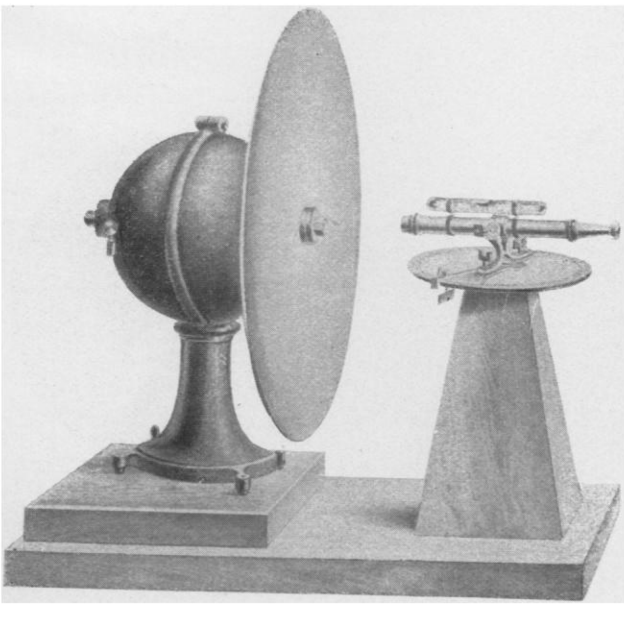

Fig. 5.

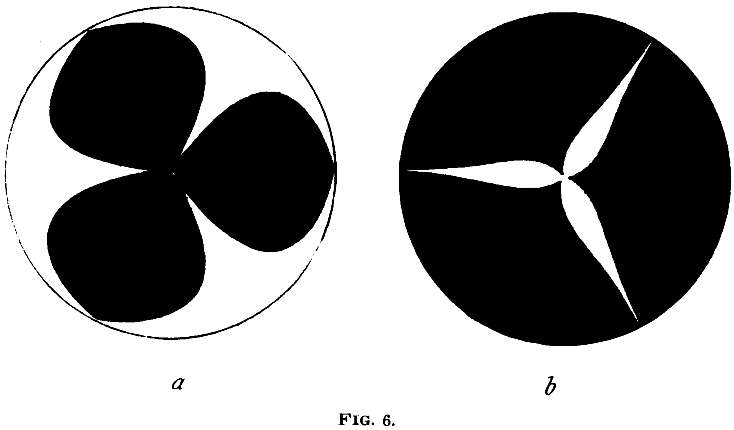

Fig. 6.

side of the disc as the original position or on the opposite side. In effecting the re-adjustment the observer will make an error of greater or less magnitude. In a great number of single experiments these errors must be ascertained for a sufficient number of normal positions. And after being trigonometrically transformed from angular values into degrees of saturation, the results must be treated according to the method of average errors, with the end in view of testing the validity of Weber's law for saturation-quantities. Experiments of this nature have been performed during the past three academic years in the Psychological Laboratory of the University of Toronto. For the further details of this investigation, I beg to refer the reader to an article on this subject which will appear in a later number of this JOURNAL.

The apparatus above described could very well be used also for experiments on the validity of the psychophysical law for light intensities. The discs necessary for this purpose must consist simply of black and white. That the intensity of light-sensation does not increase proportionally to the quantity of the stimulus, can be directly demonstrated by the appearance of these discs when in rotation. For no matter whether the white forms the leaf or the ground, the disc never makes the impression of a uniform or arithmetical transition between the two extremes; it always looks too white. If the ground is white, the disc seems to be white, with a darker spot in the centre. If the leaf is white, the periphery only seems to be of considerable darkness. In the former case the medium gray is too near the centre, in the latter too near the circumference. This state of affairs suggests the construction of a disc, where the increase or decrease of intensity follows exactly the law of Weber. In this case the outlines of the trifolium, referred to polar coördinates, are transcendental curves analogous to logarithmic curves in ordinary rectangular coördinates. In the following we may give the simple deduction of the equation for the curves concerned.

If we wish to have an increase of the intensity in a geometrical progression from the centre to the circumference, the following conditions have to be satisfied:

$$\text{When } r = a,\ \phi = \alpha;$$
$$\text{when } r = na,\ \phi = \alpha^n;$$

$$\text{from which follows that } \phi = \alpha^{\frac{r}{a}}, \quad (1)$$

$$\text{or } r = a\frac{\log \varphi.}{\log \alpha} \quad (2)$$

In order to take into account the desired size of the disc,

i. e., in order to give ϕ a determined value X at a certain distance from the centre, we have to satisfy the condition that

$$r = R, \text{ when } \varphi = X,$$

where R is the desired radius of the disc.

$$\text{We have, then, } X = a^{\frac{R}{a}},$$

$$\text{or } \log X = \frac{R \log a}{a},$$

$$\text{from which follows } \frac{a}{\log a} = \frac{R}{\log X}.$$

If we substitute this value for $\frac{a}{\log a}$ in the equations (1) and (2), we obtain

$$r = \frac{R \log \phi}{\log X},$$

$$\text{and } \phi = \sqrt[R]{X^r}. \quad (3)$$

If, on the other hand, a decrease of intensity from the centre is desired, a deduction similar to that above stated leads from the condition: when $r = na$, $\varphi = \sqrt[n]{a}$ to the equation

$$\phi = \sqrt[r]{X^R}, \qquad (4)$$

where R denotes, as above, the radius of the disc and X the desired angular value of the white at the circumference.

Fig. 7 shows schematically the geometrical construction of the curve for formula (3), *i. e.*, for a disc with intensity increasing from the centre. A denotes the black, B the white part. Fig. 6 represents two discs of this kind as they are actually used in the laboratory. For the sake of an easier blending of the components, the construction is also made here for every third of the circle. In Fig. 6 *a* the intensity increases from centre to periphery; in Fig. 6 *b* a decrease of intensity from centre to circumference takes place. Both discs, when in rotation, present to the eye a surface with apparently uniform transition from black to white. They make, to a very satisfactory degree, the impression of an arithmetical increase of intensity, and form thus an excellent means of demonstrating in a lecture in a very brief and simple manner the significance of the psychophysical law. The original curve for Fig. 6 *b* will coincide with the construction which Delbœuf gives as *construction d'une echelle des sensations.*[1]

In the above given deduction of the equations, we have

[1] J. Delbœuf, "*Etude psychophysique*," p. 92.

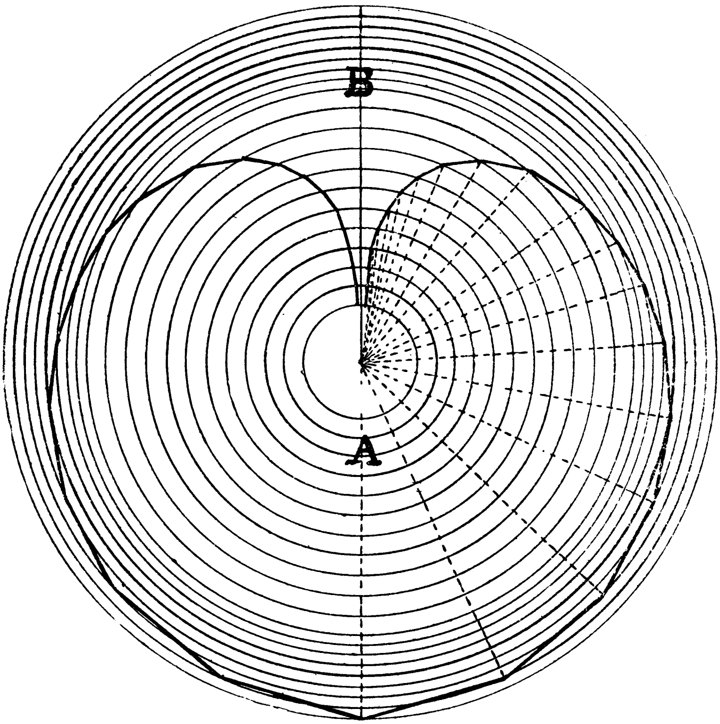

FIG. 7.

assumed that the intensity of the black was zero. This is practically only approximately secured when the black parts of the disc are made of good black velvet, or when the white parts are freely rotating before an open, completely dark space. As soon as we use material of measurable reflecting power, as, *e. g.*, black paper or lamp-black, etc., we must no longer neglect its intensity. If the reflection of the black is to that of the white as 1 is to k, the general expression for the condition, which must be satisfied at each point of the disc, is:

$$\text{When } r = na,\ k\phi + (180 - \phi) = a^n,$$

$$\text{or } \phi\,(k - 1) + 180 = a^n,$$

which, treated as in the simpler case above, leads to the equation

$$\phi\,(k - 1) + 180 = \sqrt[R]{[X\,(k - 1) + 180]^r},$$

$$\text{or } \phi = \frac{\sqrt[R]{[X\,(k - 1) + 180]^r} - 180.}{k - 1}$$

It may be mentioned that, by means of both kinds of discs, namely, those with the arithmetical and those with the logarithmic order of intensities, experiments are being carried on in our laboratory, which will be reported later. It will be easily seen that the logarithmic arrangement can just as well be applied to colored discs. The curves which separate the white and black on the uncolored part will then also be curves of the nature represented by equations (3) and (4).